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SOME OPEN PROBLEMS CONCERNING THE CONVERGENCE OF POSITIVE SERIES*

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Abstract

We discuss some old results due to Abel and Olivier concerning the convergence of positive series and prove a set of necessary conditions involving convergence in density.

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1 Introduction

Understanding the nature of a series is usually a difficult task. The following two striking examples can be found in Hardy's book (17), *Orders of infinity*: the series

$$\sum_{n\geq 3}\frac{1}{n\ln n\,(\ln\ln n)^2}$$

converges to 38.43..., but does it so slow that one needs to sum up its first $10^{3.14\times 10^{86}}$ terms to get the first two exact decimals of the sum. In the same time, the series

$$\sum_{n\geq 3}\frac{1}{n\ln n\,(\ln\ln n)}$$

is divergent but its partial sums exceed 10 only after $10^{10^{100}}$ terms. See (17), pp. 60-61. On page 48 of the same book, Hardy mentions an interesting

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result (attributed to De Morgan and Bertrand) about the convergence of the series of the form

$$\sum_{n\geq 1}\frac{1}{n^s} \text{ and } \sum_{n\geq n_k}\frac{1}{n\,(\ln n)\,(\ln\ln n)\cdots(\underbrace{\ln\ln\cdots\ln n}_{k \text{ times}})^s}\ , \qquad (MB_k)$$

where k is an arbitrarily fixed natural number, s is a real number and n_k is a number large enough to ensure that $\underbrace{\ln\ln\cdots\ln n}_{k \text{ times}}$ is positive. Precisely, such a series is convergent if $s > 1$ and divergent otherwise. This is an easy consequence of Cauchy's condensation test (see Knopp (21), p. 122). Another short argument is provided by Hardy (18) in his *Course of Pure Mathematics*, on p. 376.

The above discussion makes natural the following problem.

Problem 1. *What decides if a positive series is convergent or divergent?*

Is there any universal convergence test? Is there any pattern in convergence?

This is an old problem which received a great deal of attention over the years. Important progress was made during the 19th Century by people like A.-L. Cauchy, N. H. Abel, C. F. Gauss, A. Pringsheim and Paul du Bois-Reymond.

In 1914, Herman Müntz (24) established an unexpected connection between approximation theory and the divergence of series. Precisely, if $\lambda_0 = 0 < \lambda_1 < \lambda_2 < \cdots$ is an increasing sequence, then the vector space generated by the monomials x^{λ_k} is a dense subset of $C\left([0,1],\mathbb{R}\right)$ if and only if $\sum_{k=1}^{\infty}\frac{1}{\lambda_k} = \infty$.

In the last fifty years the interest shifted toward combinatorial aspects of convergence/divergence, although papers containing new tests of convergence continue to be published. See for example (2) and (23). This paper's purpose is to discuss the relationship between the convergence of a positive series and the convergence properties of the summand sequence.

2 Some history

We start by recalling an episode from the beginning of Analysis, that marked the moment when the series of type (MB_k) entered the attention of mathematicians. M. Goar (14) has written the story in more detail.

In 1827, L. Olivier (28) published a paper claiming that the harmonic series represents a kind of "boundary" case with which other potentially convergent series of positive terms could be compared. Specifically, he asserted that a positive series $\sum a_n$ whose terms are monotone decreasing is convergent if and only if $na_n \to 0$. One year later, Abel (1) disproved this convergence test by considering the case of the (divergent) positive series $\sum_{n\geq 2} \frac{1}{n \ln n}$. In the same *Note*, Abel (1) noticed two other important facts concerning the convergence of positive series:

Lemma 1. *There is no positive function φ such that a positive series $\sum a_n$ whose terms are monotone decreasing is convergent if and only if $\varphi(n)a_n \to 0$. In other words, there is no "boundary" positive series.*

Lemma 2. *If $\sum a_n$ is a divergent positive series, then the series $\sum \left(\frac{a_n}{\sum_{k=1}^{n} a_k} \right)$ is also divergent. As a consequence, for each divergent positive series there is always another one which diverges slower.*

A fact which was probably known to Abel (although it is not made explicit in his *Note*) is that the whole scale of divergent series

$$\sum_{n\geq n_k} \frac{1}{n\,(\ln n)\,(\ln\ln n)\cdots(\underbrace{\ln\ln\cdots\ln n}_{k \text{ times}})} \quad \text{for } k = 1, 2, 3, ... \tag{A}$$

comes from the harmonic series $\sum \frac{1}{n}$, by successive application of Lemma 2 and the following result on the generalized Euler's constant.

Lemma 3. (*C. Maclaurin and A.-L. Cauchy*). *If f is positive and strictly decreasing on $[0, \infty)$, there is a constant $\gamma_f \in (0, f(1)]$ and a sequence $(E_f(n))_n$ with $0 < E_f(n) < f(n)$, such that*

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + \gamma_f + E_f(n) \tag{MC}$$

for all n.

See (4), Theorem 1, for details.

If $f(n) \to 0$ as $n \to \infty$, then (MC) implies

$$\sum_{k=1}^{n} f(k) - \int_1^n f(x)\,dx \to \gamma_f.$$

γ_f is called the *generalized Euler's constant*, the original corresponding to $f(x) = 1/x$.

Coming back to Olivier's test of convergence, we have to mention that the necessity part survived the scrutiny of Abel and became known as Olivier's Theorem:

Theorem 1. *If $\sum a_n$ is a convergent positive series and $(a_n)_n$ is monotone decreasing, then $na_n \to 0$.*

Remark 1. *If $\sum a_n$ is a convergent positive series and $(na_n)_n$ is monotone decreasing, then $(n \ln n)\, a_n \to 0$. In fact, according to the well known estimate of harmonic numbers,*

$$\sum_1^n \frac{1}{k} = \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{\varepsilon_n}{120n^4},$$

where $\varepsilon_n \in (0,1)$, we get

$$\sum_{\lfloor\sqrt{n}\rfloor}^n a_k = \sum_{\lfloor\sqrt{n}\rfloor}^n (ka_k)\frac{1}{k} \geq na_n \sum_{\lfloor\sqrt{n}\rfloor}^n \frac{1}{k} \geq \frac{1}{2}(n \ln n)\, a_n - \frac{1}{2(\lfloor\sqrt{n}\rfloor - 1)}$$

for all $n \geq 2$. Here $\lfloor x \rfloor$ denotes the largest integer that does not exceeds x.

Simple examples show that the monotonicity condition is vital for Olivier's Theorem. See the case of the series $\sum a_n$, where $a_n = \frac{\ln n}{n}$ if n is a square, and $a_n = \frac{1}{n^2}$ otherwise.

The next result provides an extension of the Olivier's Theorem to the context of complex numbers.

Theorem 2. *Suppose that $(a_n)_n$ is a nonincreasing sequence of positive numbers converging to 0 and $(z_n)_n$ is a sequence of complex numbers such that the series $\sum a_n z_n$ is convergent. Then*

$$\lim_{n\to\infty}\left(\sum_{k=1}^n z_k\right) a_n = 0.$$

Proof. Let $\varepsilon > 0$ arbitrarily fixed. Since the series $\sum a_n z_n$ is convergent, one can choose a natural number $m > 0$ such that

$$\left|\sum_{k=m+1}^n a_k z_k\right| < \frac{\varepsilon}{4}$$

for every $n \geq m+1$. We will estimate $a_n(z_{m+1} + \cdots + z_n)$ by using Abel's identity. In fact, letting

$$S_n = a_{m+1}z_{m+1} + \cdots + a_n z_n \quad \text{for } n \geq m+1,$$

we get

$$\begin{aligned}
|a_n(z_{m+1}+\cdots+z_n)| &= a_n\left|\frac{1}{a_{m+1}}a_{m+1}z_{m+1}+\cdots+\frac{1}{a_n}a_nz_n\right| \\
&= a_n\left|\frac{1}{a_{m+1}}S_{m+1}+\frac{1}{a_{m+2}}(S_{m+2}-S_{m+1})+\cdots+\frac{1}{a_n}(S_n-S_{n-1})\right| \\
&= a_n\left|\left(\frac{1}{a_{m+1}}-\frac{1}{a_{m+2}}\right)S_{m+1}+\cdots+\left(\frac{1}{a_{n-1}}-\frac{1}{a_n}\right)S_{n-1}+\frac{1}{a_n}S_n\right| \\
&\leq \frac{\varepsilon a_n}{4}\left(\left(\frac{1}{a_{m+2}}-\frac{1}{a_{m+1}}\right)+\cdots+\left(\frac{1}{a_n}-\frac{1}{a_{n-1}}\right)+\frac{1}{a_n}\right) \\
&= \frac{\varepsilon a_n}{4}\left(\frac{2}{a_n}-\frac{1}{a_{m+1}}\right)<\frac{\varepsilon}{2}.
\end{aligned}$$

Since $\lim_{n\to\infty} a_n = 0$, one may choose an index $N(\varepsilon) > m$ such that

$$|a_n(z_1+\cdots+z_m)|<\frac{\varepsilon}{2}$$

for every $n > N(\varepsilon)$ and thus

$$|a_n(z_1+\cdots+z_n)|\leq|a_n(z_1+\cdots+z_m)|+|a_n(z_{m+1}+\cdots+z_n)|<\varepsilon$$

for every $n > N(\varepsilon)$. □

In 2003, T. Šalát and V. Toma (29) made the important remark that the monotoni-city condition in Theorem 1 can be dropped if the convergence of $(na_n)_n$ is weakened:

Theorem 3. *If $\sum a_n$ is a convergent positive series, then $na_n \to 0$ in density.*

In order to explain the terminology, recall that a subset A of $\mathbb{N}$ has *zero density* if

$$d(A)=\lim_{n\to\infty}\frac{\#(A\cap\{1,\ldots,n\})}{n}=0,$$

positive lower density if

$$\underline{d}(A)=\liminf_{n\to\infty}\frac{\#(A\cap\{1,\ldots,n\})}{n}>0,$$

and *positive upper density* if

$$\bar{d}(A)=\limsup_{n\to\infty}\frac{\#(A\cap\{1,\ldots,n\})}{n}>0.$$

Here # stands for cardinality.

We say that a sequence $(x_n)_n$ of real numbers *converges in density* to a number x (denoted by (d)-$\lim_{n\to\infty} x_n = x$) if for every $\varepsilon > 0$ the set

$A(\varepsilon) = \{n : |x_n - x| \geq \varepsilon\}$ has zero density. Notice that $(d) - \lim_{n\to\infty} x_n = x$ if and only if there is a subset J of $\mathbb{N}$ of zero density such that

$$\lim_{\substack{n\to\infty \\ n\notin J}} a_n = 0.$$

This notion can be traced back to B. O. Koopman and J. von Neumann ((22), pp. 258-259), who proved the integral counterpart of the following result:

Theorem 4. *For every sequence of nonnegative numbers,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k = 0 \Rightarrow (d)\text{-} \lim_{n\to\infty} a_n = 0.$$

The converse works under additional hypotheses, for example, for bounded sequences.

Proof. Assuming $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k = 0$, we associate to each $\varepsilon > 0$ the set $A_\varepsilon = \{n \in \mathbb{N} : a_n \geq \varepsilon\}$. Since

$$\begin{aligned} \frac{|\{1, ..., n\} \cap A_\varepsilon|}{n} &\leq \frac{1}{n} \sum_{k=1}^{n} \frac{a_k}{\varepsilon} \\ &\leq \frac{1}{\varepsilon n} \sum_{k=1}^{n} a_k \to 0 \text{ as } n \to \infty, \end{aligned}$$

we infer that each of the sets A_ε has zero density. Therefore (d)-$\lim_{n\to\infty} a_n = 0$.

Suppose now that $(a_n)_n$ is bounded and (d)-$\lim_{n\to\infty} a_n = 0$. Then for every $\varepsilon > 0$ there is a set J of zero density outside which $a_n < \varepsilon$. Since

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{n} a_k &= \frac{1}{n} \sum_{k\in\{1,...,n\}\cap J} a_k + \frac{1}{n} \sum_{k\in\{1,...,n\}\setminus J} a_k \\ &\leq \frac{|\{1, ..., n\} \cap J|}{n} \cdot \sup_{k\in\mathbb{N}} a_k + \varepsilon \end{aligned}$$

and $\lim_{n\to\infty} \frac{|\{1,...,n\}\cap J|}{n} = 0$, we conclude that $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k = 0$. □

Remark 2. *Theorem 4 is related to the Tauberian theory, whose aim is to provide converses to the well known fact that for any sequence of complex numbers,*

$$\lim_{n\to\infty} z_n = z \Rightarrow \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} z_k = z.$$

Recall here the famous Hardy-Littlewood Tauberian theorem: If $|z_n - z_{n-1}| = O(1/n)$ *and*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} z_k = z,$$

then $\lim_{n\to\infty} z_n = z$. *See (19), Theorem 28.*

The aforementioned result of Šalát and Toma is actually an easy consequence of Theorem 4. Indeed, if $\sum a_n$ is a convergent positive series, then its partial sums $S_n = \sum_{k=1}^{n} a_k$ constitute a convergent sequence with limit S. By Cesàro's Theorem,

$$\lim_{n\to\infty} \frac{S_1 + \cdots + S_{n-1}}{n} = S,$$

whence

$$\lim_{n\to\infty} \frac{a_1 + 2a_2 + \cdots + na_n}{n} = \lim_{n\to\infty}\left(S_n - \frac{S_1 + \cdots + S_{n-1}}{n}\right) = 0.$$

According to Theorem 4, this fact is equivalent to the convergence in density of $(na_n)_n$ to 0.

In turn, the result of Šalát and Toma implies Olivier's Theorem. Indeed, if the sequence (a_n) is decreasing, then

$$\frac{a_1 + 2a_2 + \cdots + na_n}{n} \geq \frac{(1 + 2 + \cdots + n)a_n}{n} = \frac{(n+1)a_n}{2}$$

which implies that if

$$\lim_{n\to\infty} \frac{a_1 + 2a_2 + \cdots + na_n}{n} = 0$$

then $\lim_n na_n = 0$.

If $\sum a_n$ is a convergent positive series, then so is $\sum a_{\varphi(n)}$, whenever $\varphi : \mathbb{N} \to \mathbb{N}$ is a bijective map. This implies that $na_{\varphi(n)} \to 0$ in density (a conclusion that doesn't work for usual convergence).

The monograph of H. Furstenberg (13) outlines the importance of convergence in density in ergodic theory. In connection to series summation, the concept of convergence in density was rediscovered (under the name of statistical convergence) by Steinhaus (30) and Fast (12) (who mentioned also the first edition of Zygmund's monograph (33), published in Warsaw in 1935). Apparently unaware of the Koopman-von Neumann result, Šalát and Toma referred to these authors for the roots of convergence in density.

At present there is a large literature about this concept and its many applications. We only mention here the recent papers by M. Burgin and O. Duman (7) and P. Therán (32).

3 An extension of Šalát - Toma Theorem

In this section we will turn our attention toward a generalization of the result of Šalát and Toma mentioned above. This generalization involves the concepts of convergence in density and convergence in lower density. A sequence $(x_n)_n$ of real numbers *converges in lower density* to a number x (abbreviated, $(\underline{d})$-$\lim_{n\to\infty} x_n = x$) if for every $\varepsilon > 0$ the set $A(\varepsilon) = \{n : |x_n - x| \geq \varepsilon\}$ has zero lower density.

Theorem 5. *Assume that $\sum a_n$ is a convergent positive series and $(b_n)_n$ is a nondecreasing sequence of positive numbers such that $\sum_{n=1}^{\infty} \frac{1}{b_n} = \infty$. Then*

$$(\underline{d})\text{-}\lim_{n\to\infty} a_n b_n = 0,$$

and this conclusion can be improved to

$$(d)\text{-}\lim_{n\to\infty} a_n b_n = 0,$$

provided that $\inf_n \frac{n}{b_n} > 0$.

An immediate consequence is the following result about the speed of convergence to 0 of the general term of a convergent series of positive numbers.

Corollary 1. *If $\sum a_n$ is a convergent series of positive numbers, then for each $k \in \mathbb{N}$,*

$$(\underline{d})\text{ - }\lim_{n\to\infty} \left[n\,(\ln n)\,(\ln\ln n)\cdots(\underbrace{\ln\ln\cdots\ln n}_{k\ times})a_n \right] = 0. \qquad (\underline{D}_k)$$

The proof of Theorem 5 is based on two technical lemmas:

Lemma 4. *Suppose that $(c_n)_n$ is a nonincreasing sequence of positive numbers such that $\sum_{n=1}^{\infty} c_n = \infty$ and S is a set of positive integers with positive lower density. Then the series $\sum_{n\in S} c_n$ is also divergent.*

Proof. By our hypothesis there are positive integers p and N such that

$$\frac{|S \cap \{1, ..., n\}|}{n} > \frac{1}{p}$$

whenever $n \geq N$. Then $|S \cap \{1, ..., kp\}| > k$ for every $k \geq N/p$, which yields

$$\begin{aligned}
\sum_{n\in S} c_n &= \sum_{k=1}^{\infty} c_{n_k} \geq \sum_{k=1}^{\infty} c_{kp} \\
&= \frac{1}{p} \sum_{k=1}^{\infty} p c_{kp} \\
&\geq \frac{1}{p} \left[(c_p + \cdots + c_{2p-1}) + (c_{2p} + \cdots + c_{3p-1}) + \cdots \right] \\
&= \frac{1}{p} \sum_{k=p}^{\infty} c_k = \infty.
\end{aligned}$$

□

Our second lemma shows that a subseries $\sum_{n\in S}\frac{1}{n}$ of the harmonic series is divergent whenever S is a set of positive integers with positive upper density.

Lemma 5. *If S is an infinite set of positive integers and $(a_n)_{n\in S}$ is a nonincreasing positive sequence such that $\sum_{n\in S} a_n < \infty$ and $\inf\{nc_n : n \in S\} = \alpha > 0$, then S has zero density.*

Proof. According to our hypotheses, the elements of S can be counted as $k_1 < k_2 < k_3 < \ldots$. Since

$$0 < \frac{n}{k_n} = \frac{na_{k_n}}{k_n a_{k_n}} \le \frac{1}{\alpha} n a_{k_n},$$

we infer from Theorem 1 that $\lim_{n\to\infty}\frac{n}{k_n} = 0$. Then

$$\frac{|S\cap\{1,...,n)|}{n} = \frac{p}{n} = \frac{|S\cap\{1,...,k_p)|}{k_p} \le \frac{p}{k_p} \to 0,$$

whence

$$d(S) = \lim_{n\to\infty}\frac{|S\cap\{1,...,n)|}{n} = 0.$$

□

Proof of Theorem 5. For $\varepsilon > 0$ arbitrarily fixed we denote

$$S_\varepsilon = \{n : a_n b_n \ge \varepsilon\}.$$

Then

$$\infty > \sum\nolimits_{n\in S_\varepsilon} a_n \ge \sum\nolimits_{n\in S_\varepsilon}\frac{1}{b_n},$$

whence by Lemma 4 it follows that S_ε has zero lower density. Therefore $(\underline{d})$-$\lim_{n\to\infty} a_n b_n = 0$. When $\inf_n \frac{n}{b_n} = \alpha > 0$, then

$$\infty > \sum\nolimits_{n\in S_\varepsilon}\frac{1}{b_n} \ge \alpha\sum\nolimits_{n\in S_\varepsilon}\frac{1}{n}$$

so by Lemma 5 we infer that S_ε has zero density. In this case, (d)-$\lim_{n\to\infty} a_n b_n = 0$. □

4 Convergence associated to higher order densities

The convergence in lower density is very weak. A better way to formulate higher order Šalát-Toma type criteria is to consider the convergence in harmonic density. We will illustrate this idea by proving a non-monotonic version of Remark 1.

The harmonic density d_h is defined by the formula

$$d_h(A) = \lim_{n\to\infty} \frac{1}{\ln n} \sum_{k=1}^{n} \frac{\chi_A(k)}{k},$$

and the limit in harmonic density, $(d_h)\text{-}\lim_{n\to\infty} a_n = \ell$, means that each of the sets $\{n : |a_n - \ell| \geq \varepsilon\}$ has zero harmonic density, whenever $\varepsilon > 0$. Since

$$d(A) = 0 \text{ implies } d_h(A) = 0,$$

(see (16), Lemma 1, p. 241), it follows that the existence of limit in density assures the existence of limit in harmonic density.

The harmonic density has a nice application to Benford's law, which states that in lists of numbers from many real-life sources of data the leading digit is distributed in a specific, non-uniform way. See (8) for more details.

Theorem 6. *If* $\sum a_n$ *is a convergent positive series, then*

$$(d_h)\text{-}\lim_{n\to\infty} (n \ln n)\, a_n = 0.$$

Proof. We start by noticing the following analogue of Lemma 5: If $(b_n)_n$ is a positive sequence such that $(nb_n)_n$ is decreasing and

$$\inf (n \ln n)\, b_n = \alpha > 0,$$

then every subset S of $\mathbb{N}$ for which $\sum_{n\in S} b_n < \infty$ has zero harmonic density.

To prove this assertion, it suffices to consider the case where S is infinite and to show that

$$\lim_{x\to\infty} \left(\sum\nolimits_{k\in S\cap\{1,\ldots,n\}} \frac{1}{k} \right) nb_n = 0. \tag{H}$$

The details are very similar to those used in Lemma 5, and thus they are omitted.

Having (H) at hand, the proof of Theorem 6 can be completed by considering for each $\varepsilon > 0$ the set

$$S_\varepsilon = \{n : (n \ln n)\, a_n \geq \varepsilon\}.$$

Since

$$\varepsilon \sum\nolimits_{n\in S_\varepsilon} \frac{1}{n \ln n} \leq \sum\nolimits_{n\in S_\varepsilon} a_n < \infty,$$

by the aforementioned analogue of Lemma 5 applied to $b_n = 1/(n \ln n)$ we infer that S_ε has zero harmonic density. Consequently $(d_h)\text{-}\lim_{x\to\infty} (n \ln n)\, a_n = 0$, and the proof is done. $\square$

An integral version of the previous theorem can be found in (25) and (26).

One might think that the fulfilment of a sequence of conditions like (D_k), for all $k \in \mathbb{N}$, (or something similar) using other series, is strong enough to force the convergence of a positive series $\sum a_n$. That this is not the case was shown by Paul du Bois-Raymond (6) (see also (21), Ch. IX, Section 41) who proved that for every sequence of divergent positive series, each divergent essentially slower than the previous one, it is possible to construct a series diverging slower than all of them.

Under these circumstances the following problem seems of utmost interest:

Problem 2. *Find an algorithm to determine whether a positive series is convergent or not.*

5 The relevance of the harmonic series

Surprisingly, the study of the nature of positive series is very close to that of subseries of the harmonic series $\sum \frac{1}{n}$.

Lemma 6. *If $(a_n)_n$ is an unbounded sequence of real numbers belonging to $[1, \infty)$, then the series $\sum \frac{1}{a_n}$ and $\sum \frac{1}{\lfloor a_n \rfloor}$ have the same nature.*

Proof. This follows from the Comparison Test and the inequality $\lfloor x \rfloor \le x < 2\lfloor x \rfloor$, which works for every $x \ge 1$. $\square$

By combining Lemma 5 and Lemma 6 we infer the following result:

Corollary 2. *If $(a_n)_n$ is a sequence of positive numbers whose integer parts form a set of positive upper density, then the series $\sum \frac{1}{a_n}$ is divergent.*

The converse of Corollary 2 is not true. A counterexample is provided by the series $\sum_{p=\text{prime}} \frac{1}{p}$, of inverses of prime numbers, which is divergent (see (3) or (10) for a short argument). According to an old result due to Chebyshev, if $\pi(n) = |\{p \le n : p \text{ prime}\}|$, then

$$\frac{7}{8} < \frac{\pi(n)}{n/\ln n} < \frac{9}{8}$$

and thus the set of prime numbers has zero density.

The following estimates of the kth prime number,

$$k\,(\ln k + \ln\ln k - 1) \le p_k \le k\,(\ln k + \ln\ln k) \quad \text{for } k \ge 6,$$

which are made available by a recent paper of P. Dusart (9), show that the speed of divergence of the series $\sum_{p=\text{prime}} \frac{1}{p}$ is comparable with that of $\sum \frac{1}{k(\ln k + \ln\ln k)}$.

Lemma 6 suggests that the nature of positive series $\sum \frac{1}{a_n}$ could be related to some combinatorial properties of the sequence $(\lfloor a_n \rfloor)_n$ (of natural numbers).

Problem 3. *Given an increasing function* $\varphi : \mathbb{N} \to (0, \infty)$ *with* $\lim_{n\to\infty} \varphi(n) = \infty$, *we define the upper density of weight* φ *by the formula*

$$\bar{d}_\varphi(A) = \limsup_{n\to\infty} \frac{|A \cap [1, n]|}{\varphi(n)}.$$

Does every subset $A \subset \mathbb{N}$ *with* $\bar{d}_{n/\ln n}(A) > 0$ *generate a divergent subseries* $\sum_{n\in A} \frac{1}{n}$ *of the harmonic series?*

What about the case of other weights

$$n/[(\ln n)\,(\ln \ln n) \cdots (\underbrace{\ln \ln \cdots \ln n}_{k\ times})]?$$

This problem seems important in connection with the following longstanding conjecture due to P. Erdös:

Conjecture 1. (*P. Erdös*). *If the sum of reciprocals of a set A of integers diverges, then that set contains arbitrarily long arithmetic progressions.*

This conjecture is still open even if one only seeks a single progression of length three. However, in the special case where the set A has positive upper density, a positive answered was provided by E. Szemerédi (31) in 1975. Recently, Green and T. Tao (15) proved Erdös' Conjecture in the case where A is the set of prime numbers, or a relatively dense subset thereof.

Theorem 7. *Assuming the truth of Erdös' conjecture, any unbounded sequence* $(a_n)_n$ *of positive numbers whose sum of reciprocals* $\sum_n \frac{1}{a_n}$ *is divergent must contain arbitrarily long* ε*-progressions, for any* $\varepsilon > 0$.

By an ε-progression of length n we mean any string $c_1, ..., c_n$ such that

$$|c_k - a - kr| < \varepsilon$$

for suitable $a, r \in \mathbb{R}$ and all $k = 1, ..., n$.

The converse of Theorem 7 is not true. A counterexample is provided by the convergent series $\sum_{n=1}^{\infty} \left(\frac{1}{10^n+1} + \cdots + \frac{1}{10^n+n} \right)$.

It seems to us that what is relevant in the matter of convergence is not only the existence of some progressions but the number of them. We believe not only that the divergent subseries of the harmonic series have progressions of arbitrary length but that they have a huge number of such progressions and of arbitrarily large common differences. Notice that the counterexample above contains only progressions of common difference 1 (or subprogressions of them). Hardy and Littlewood's famous paper (20) advanced the hypothesis that the number of progressions of length k is asymptotically of the form $C_k n^2/\ln^k n$, for some constant C_k.

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